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Particle representations for SPDEs with applications

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Outline

1. Weighted particle representation
2. System by atomic measure
3. Poisson particle representation
4. Application to stochastic filtering
5. Application to FBSDEs
6. Application to superprocesses

1. Weighted particle representation

SPDE we shall study

$$\frac{\partial V(t,x)}{\partial t} = L_2(V(t))V(t,x) + \int_U L_1(V(t),u)V(t,x)\frac{W(dudt)}{dt} \quad (1.1)$$

$W(duds)$ noise in space-time
$L_1(V(t),u)$ first order diff. operator
$L_2(V(t))$ second order diff. operator

$$\begin{aligned}\langle \phi, V(t)\rangle &= \langle \phi, V(0)\rangle + \int_0^t \langle L_2(V(s))\phi, V(s)\rangle\, ds \\ &\quad + \int_{U\times[0,t]} \langle L_1(V(s),u)\phi, V(s)\rangle\, W(duds)\end{aligned} \tag{1.2}$$

where

$$L_1(\nu,u)\phi(x) = \beta(x,\nu,u)\phi(x) + \alpha^T(x,\nu,u)\nabla\phi(x),$$

$$\begin{aligned}L_2(\nu)\phi(x) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,\nu)\partial_{x_i}\partial_{x_j}\phi(x) \\ &\quad + b(x,\nu)^*\nabla\phi(x) + d(x,\nu)\phi(x).\end{aligned}$$

Choose σ and γ s.t.

$$\begin{aligned} a(x,\nu) &= \sigma(x,\nu)\sigma^T(x,\nu) \\ &\quad + \int_U \alpha(x,\nu,u)\alpha^T(x,\nu,u)\mu(du) \end{aligned}$$

and

$$\begin{aligned} b(x,\nu) &= c(x,\nu) + \sigma(x,\nu)\gamma(x,\nu) \\ &\quad + \int_U \beta(x,\nu,u)\alpha(x,\nu,u)\mu(du) \end{aligned}$$

Consider system with locations

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V(s))dB_i(s) + \int_0^t c(X_i(s), V(s))ds + \int_{U\times[0,t]} \alpha(X_i(s), V(s), u)W(duds) \tag{1.3}$$

with

$$V(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} A_i(t)\delta_{X_i(t)} \tag{1.4}$$

and weights

$$
\begin{aligned}
& A_i(t) \\
= \;& A_i(0) + \int_0^t A_i(s)\gamma^T(X_i(s), V(s))dB_i(s) \\
& + \int_0^t A_i(s)d(X_i(s), V(s))ds \\
& + \int_{U\times[0,t]} A_i(s)\beta(X_i(s), V(s), u)W(duds).
\end{aligned} \tag{1.5}
$$

Theorem

$V(t)$ is sol. to (1.4) iff it is a sol. to SPDE (1.2).

Most with constant weight 1:

McKean ('67)

$\left\{\begin{array}{l}\text{Hitsuda \& Mitoma ('86) conti.}\\ \text{Graham ('96) jump}\end{array}\right.$

∞-dim. $\left\{\begin{array}{l}\text{Chiang, Kallianpur \& Sundar ('91), Cont.}\\ \text{Kallianpur \& Xiong ('94) jump}\\ \text{Xiong ('95)}\end{array}\right.$

non-constant weights $\left\{\begin{array}{l}\text{Non-random: Dawson \& Vaillancourt ('95)}\\ \text{Random: Kotelenez ('95) } A_i(t)\equiv A_i(0)\end{array}\right.$

Conditions

$$|\sigma(x,\nu)|^2 \leq K^2$$

$$|\sigma(x_1,\nu_1) - \sigma(x_2,\nu_2)|^2 \leq K^2(|x_1 - x_2|^2 + \rho(\nu_1,\nu_2)^2)$$

where

$$\rho(\nu_1,\nu_2) = \sup\{|\langle \phi, \nu_1 - \nu_2 \rangle| : \|\phi\|_\infty \leq 1, Lip(1)\}.$$

Similar conditions for other coeff.

Theorem

The system (1.3-1.5) has a solution.

Idea of proof

Sequence of approximation

$$B_i^n(t) = B_i\left(\frac{[nt]}{n}\right), \qquad D_n(t) = \frac{[nt]}{n}$$

$$W^n(A \times [0,t]) = W\left(A \times [0, \frac{[nt]}{n}]\right)$$

Solution (X^n, A^n, V^n) weak convergence to (X, A, V)

Theorem

The system (1.3-1.5) has at most one solution.

Idea of proof
Uniqueness of system:
Lipschitz condition is not satisfied: e.g. $(a, x) \mapsto a\beta(x)$
Local Lipschitz, i.e. on

$$\{(a_i) : |a_i| \leq M, \ i = 1, 2, \cdots\}$$

How to truncate?

Theorem

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Uniqueness of system:

Lipschitz condition is not satisfied: e.g. $(a, x) \mapsto a\beta(x)$

Local Lipschitz, i.e. on

$$\{(a_i) : |a_i| \le M,\ i = 1, 2, \cdots\}$$

How to truncate?

Truncate the average:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} A_i(t)^2$$

How to prove uniqueness for SPDE (1.2)?
Consider Linear SPDE

$$\begin{aligned}\langle \phi, U(t)\rangle &= \langle \phi, U(0)\rangle + \int_0^t \langle L_{2,s}\phi, U(s)\rangle \, ds \\ &\quad + \int_{U\times[0,t]} \langle L_{1,s,u}\phi, U(s)\rangle \, W(duds) \qquad (1.6)\end{aligned}$$

where

$$L_{2,s}\phi = L_2(V(s))\phi$$

$$L_{1,s,u}\phi = L_1(V(s), u)\phi$$

Theorem

If $V_0 \in L^2(\mathbb{R}^d)$, then the linear SPDE (1.6) has at most one solution.

Key: Smooth out by $Z_\delta(t) = T_\delta U(t)$.

Theorem

If $V_0 \in L^2(\mathbb{R}^d)$, then SPDE (1.2) has at most one solution.

Proof

Let $V_1(t)$ be another solution.
Consider

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V_1(s)) dB_i(s) + \cdots$$

$$A_i(t) = \cdots$$

Let

$$V_2(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} A_i(t)\delta_{X_i(t)}$$

Then V_2 is a solution to

$$\begin{aligned} \langle \phi, U(t) \rangle &= \langle \phi, U(0) \rangle + \int_0^t \langle L_2(V_1(s))\phi, U(s) \rangle \, ds \qquad (1.7) \\ &\quad + \int_{U\times[0,t]} \langle L_1(V_1(s), u)\phi, U(s) \rangle \, W(duds). \end{aligned}$$

V_1 is also a solution to (1.7). By uniqueness, $V_1 = V_2$. V_1 is a solution to (1.3-1.5). $V_1 = V$

2. System by atomic measure

For $i = 1, 2, \cdots$, the position of the ith particle is

$$X_t^i = X_0^i + \int_0^t \sigma(X_s^i) dW_s^i + \int_0^t b(X_s^i, N_s) ds + \int_0^t \alpha(X_s^i) dW_s, \tag{2.1}$$

where

$$N_s = \sum_i \delta_{X_s^i}.$$

Condition (B): $\exists\ c,\ r > 0$ s.t., $\forall\ x,\ z^i \in \mathbb{R}$, $i = 1, 2, \cdots$, we have

$$|\sigma(x)| + |\alpha(x)| + \left| b\left(x, \sum \delta_{z^i}\right)\right| \leq K.$$

Condition (Lip): σ, α Lip., and $\exists\ c,\ r>0$ s.t., $\forall x,\ \tilde{x},\ z^i,\ \tilde{z}^i \in \mathbb{R}$, $i=1,2,\cdots$, we have

$$\begin{aligned} & \left|b\left(x,N\right)-b\left(\tilde{x},\tilde{N}\right)\right| \\ \leq\ & c|x-\tilde{x}|\left(N(S_r(x))+\tilde{N}(S_r(\tilde{x}))\right) \\ & +c\left(\sum_{|z^i-x|\leq r}|z^i-\tilde{z}^i|+\sum_{|\tilde{z}^i-\tilde{x}|\leq r}|z^i-\tilde{z}^i|\right) \qquad (2.2) \end{aligned}$$

where

$$N=\sum \delta_{z^i}.$$

Typical example

$$b(x,\nu) = \phi\left(x, \int h(x-y)\nu(dy)\right)$$

where ϕ bounded Lipschitz in both variables, h bounded Lipschitz w/ compact support.

Condition (I): $X_0^i = i$, $i \in \mathbb{Z}$.

Lemma

For any $c_1 > 2$ and $t > 0$, there exists a positive constant c_2 such that

$$\mathbb{P}\left(N_t([-n,n]) > c_1 n\right) \leq c_2 e^{-(c_1-2)n}, \qquad \forall n \in \mathbb{N}. \tag{2.3}$$

Corollary

There exists a constant c_3 such that for all n,

$$\mathbb{E}\left(N_t([-n,n])^2 1_{N_t([-n,n])>3n}\right) \leq c_2 n e^{-n}.$$

Let c be s.t.

$$\rho(a) = c1_{|a|<1} \exp\left(-1/(1-a^2)\right), \qquad a \in \mathbb{R}$$

becomes a p.d.f. Let

$$\phi(x) = \int_{\mathbb{R}} e^{-|a|}\rho(x-a)da, \qquad x \in \mathbb{R}.$$

Then, ϕ is smooth and

$$c_n e^{-|x|} \le \phi^{(n)}(x) \le C_n e^{-|x|}, \qquad \forall\, x \in \mathbb{R}.$$

Also

$$\phi(x^i) \le \phi(\tilde{x}^j) \exp\left(|x^i - \tilde{x}^i| + |\tilde{x}^j - \tilde{x}^i|\right). \tag{2.4}$$

Proposition

The system (2.1) has at most one solution.

Idea of proof Let $\{\tilde{X}_t^i,\ i \in \mathbb{Z}\}$ be another solution.

$$f(t) \equiv \mathbb{E} \sum_i |X_t^i - \tilde{X}_t^i|^2 \phi(X_t^i) \leq I_0 + I_1 + I_2 \qquad (2.5)$$

where

$$I_0 = c \int_0^t \mathbb{E} \sum_i |X_s^i - \tilde{X}_s^i|^2 \phi(X_s^i) ds,$$

$$I_1 = c \int_0^t \mathbb{E} \sum_i \sum_j |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) 1_{|X_s^j - X_s^i| \le 1} ds,$$

$$I_2 = c \int_0^t \mathbb{E} \sum_i \sum_j |X_s^j - \tilde{X}_s^j|^2 \phi(\tilde{X}_s^j) 1_{|\tilde{X}_s^j - \tilde{X}_s^i| \le 1} ds.$$

$$
\begin{aligned}
& I_1 \\
\leq\ & c\int_0^t \mathbb{E}\sum_j |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j)\#\{i:\ |X_s^j - X_s^i| \leq 1\} ds \\
\leq\ & c\int_0^t \mathbb{E}\left\{\left(\sum_{j:\ |X_s^j|<n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j)\right) N_s([-n,n])\right\} ds \\
& +c\int_0^t \mathbb{E}\left\{\sum_{j:\ |X_s^j|\geq n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j)\#\{i:\ |X_s^j - X_s^i| \leq 1\}\right\} ds \\
\equiv\ & I_{11} + I_{12}.
\end{aligned}
$$

$$I_{12} \le ce^{-n/2},$$

$$\begin{aligned} I_{11} &\le 3cn\int_0^t \mathbb{E}\sum_{j:\ |X_s^j|<n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) ds \\ &\quad + c\int_0^t \mathbb{E}\Bigg\{ \Bigg(\sum_{j:\ |X_s^j|<n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) \Bigg) \\ &\qquad \times N_s([-n,n]) 1_{N_s([-n,n])>3n} \Bigg\} ds \\ &\le cn\int_0^t f(s)ds + ce^{-\delta n}. \end{aligned}$$

Then,

$$I_1 \leq cn \int_0^t f(s)ds + 2ce^{-\delta n}.$$

Similarly

$$I_2 \leq cn \int_0^t \tilde{f}(s)ds + ce^{-\delta n}. \tag{2.6}$$

Hence,

$$f(t) \leq cn \int_0^t \left(f(s) + \tilde{f}(s) \right) ds + ce^{-\delta n}.$$

Similarly, we can prove that

$$\tilde{f}(t) \leq cn \int_0^t \left(f(s) + \tilde{f}(s) \right) ds + ce^{-\delta n}.$$

Adding two equations, we get

$$f(t) + \tilde{f}(t) \leq cn \int_0^t \left(f(s) + \tilde{f}(s) \right) ds + ce^{-\delta n}.$$

By Gronwall's inequality, for $t < \delta$, we have

$$f(t) + \tilde{f}(t) \leq ce^{-\delta n} e^{nt} \to 0$$

This gives us uniqueness for $t < \delta$.
How to get uniqueness for any t?

Adding two equations, we get

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This gives us uniqueness for $t < \delta$.
How to get uniqueness for any t?
We get uniqueness by replacing $e^{-|x|}$ by $e^{-\lambda|x|}$.

3. Poisson particle representation

Consider system: For $i = 1, 2, \cdots,$

$$X_t^i = X_0^i + \int_0^t \sigma(X^i) dW_i(s) + \int_0^t b(X_s^i, V_s) ds + \int_{U\times[0,t]} \alpha(X_s^i, u) W(duds) \tag{3.1}$$

and

$$U_t^i = U_0^i + \int_0^t U_s^i \gamma(X_s^i) dW_i(s) + \int_0^t U_s^i a(X_s^i, V_s) ds + \int_{U\times[0,t]} U_s^i \beta(X_s^i, u) W(duds), \tag{3.2}$$

where V is the measure-valued process given by

$$V(t) = \lim_{u\to\infty} \frac{1}{u} \sum_{U_t^i \le u} \delta_{X_t^i}. \tag{3.3}$$

Remark

i) Given V*,*

$$\sum_i \delta_{(X_t^i, U_t^i)}$$

is a Poisson random measure on $\mathbb{R} \times \mathbb{R}_+$ *with intensity measure* $V(t) \times m$*, where* m *is the Lebesgue measure on* $\mathbb{R}_+$*.*
ii) Advantages of Poisson representation: $V(t)$ *can be infinite measure.*
iii)

$$\begin{aligned} V(t) &= \lim_{\epsilon \to 0+} \epsilon \sum_{i=1}^{\infty} e^{-\epsilon U_t^i} \delta_{X_t^i} \\ &= \lim_{\epsilon \to 0+} \epsilon^2 \sum_{i=1}^{\infty} U_t^i e^{-\epsilon U_t^i} \delta_{X_t^i}. \end{aligned}$$

Condition (B) remains true.
Wasserstein distance by

$$\rho_x(\nu_1,\nu_2)=\sup\left\{\left|\int_{|y-x|\le 1} f(y)(\nu_1-\nu_2)(dy)\right|:\ f\in\mathbb{B}_1\right\},$$

where $\mathbb{B}_1$ is the unit ball under Lipschitz norm, i.e.,

$$\mathbb{B}_1=\{f:\mathbb{R}\to\mathbb{R},\ |f(x)|\le 1,\ |f(x)-f(y)|\le |x-y|\}.$$

Condition (Lip): $\exists\, K$ s.t. $\forall\, x,\ \tilde{x} \in \mathbb{R}$ and $\nu,\ \tilde{\nu} \in M(\mathbb{R})$, we have

$$\begin{aligned} & |b(x,\nu) - b(\tilde{x},\tilde{\nu})| \\ \leq\ & K\left(|x-\tilde{x}|\left(1+\nu(S_1(x))+\tilde{\nu}(S_1(\tilde{x}))\right)+\rho_x(\nu,\tilde{\nu})+\rho_{\tilde{x}}(\nu,\tilde{\nu})\right), \end{aligned}$$

where $S_1(x)$ denotes the interval $(x-1, x+1)$.

Suppose that (3.1-3.3) has two solutions (X_i, U_i, V) and $(\tilde{X}_i, \tilde{U}_i, \tilde{V})$, $i = 1, 2, \cdots$.

We will need the following large ball type estimate.

Lemma

Suppose that

$$\sup_{y\in\mathbb{R}}\left(V_0(S_2(y))+\int_{|x-y|>2}\frac{V_0(dx)}{|x-y-1|}\right)<\infty,$$

where $S_2(y)$ is the interval with center y and radius 2. Then, there are constants c, $\delta>0$ such that for any $s\in[0,T]$ and $n\in\mathbb{N}$, we have

$$\mathbb{P}\left(\sup_{|y|\le n}V_s(S_1(y))>\sqrt{n}\right)\le ce^{-\delta n}. \tag{3.4}$$

Key in the proof i) V solution to (1.2).
ii) By the exponential formula of Boué-Dupuis, we have

$$-\log \mathbb{E} e^{f(W)} = \inf_u \mathbb{E}\left(\frac{1}{2}\|u\|^2 - f\left(W + \int_0^\cdot u_s ds\right)\right), \qquad (3.5)$$

where the infimum is taken over all processes u that are $\mathcal{F}_t^W$-predictable such that

$$\|u\|^2 = \int_0^T u_t^2 dt < \infty, \qquad a.s.$$

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Theorem

The infinite system (3.1-3.3) has a unique solution.

Idea of proof Let

$$f(t) \equiv \lim_{\epsilon \to 0} \sum_i \epsilon e^{-\epsilon U_t^i} |X_t^i - \tilde{X}_t^i|^2 \phi(X_t^i),$$

and

$$g(t) = \lim_{\epsilon \to 0} \mathbb{E} \sum_i \epsilon^3 e^{-\epsilon U_t^i} (U_t^i - \tilde{U}_t^i)^2 \phi(X_t^i).$$

$\tilde{f}$ and $\tilde{g}$ are defined similarly.

Let $h(t)$ be the sum of $f(t)$, $g(t)$, $\tilde{f}(t)$ and $\tilde{g}(t)$. Then,

$$h(t) \leq cn \int_0^t h(s)ds + ce^{-\delta n}.$$

The uniqueness then follows from Gronwall's inequality.

4. Application to stochastic filtering

Signal process

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

Observation process

$$Y_t = \int_0^t h(X_s)ds + W_t.$$

Information

$$\mathcal{G}_t = \sigma\{Y_s : \ s \leq t\}.$$

Optimal filter

$$\langle f, \pi_t \rangle = \mathbb{E}\left(f(X_t)|\mathcal{G}_t\right).$$

Kallianpur-Striebel formula,

$$\langle f, \pi_t \rangle = \frac{\langle f, \mu_t \rangle}{\langle 1, \mu_t \rangle},$$

where

$$\langle f, \mu_t \rangle = \hat{\mathbb{E}} \left(f(X_t) M_t | \mathcal{G}_t \right)$$

and $\hat{\mathbb{E}}$ is expectation wrt $\hat{P}$ under which (B, Y) is B.M.

Zakai equation: Special case of (1.1) with $W = Y$ and

$$L_2\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i}\partial_{x_j}\phi(x) + b(x)^*\nabla\phi(x),$$

$$L_1\phi(x) = h(x)\phi(x).$$

Kushner-FKK equation: Special case of (1.1) with W being the innovation process and

$$L_1(\nu, u)\phi(x) = h(x) - \langle h, \nu \rangle .$$

5. Application to FBSDEs

Consider

$$\left\{\begin{array}{rcl} dX(t) & = & b(X(t), Y(t), Z(t))dt + \sigma(X(t), Y(t), Z(t))dW(t), \\ dY(t) & = & g(X(t), Y(t), Z(t))dt + Z(t)dW(t), \\ X(0) & = & x, \;\; Y(T) = h(X(T)). \end{array}\right.$$

How to solve FBSDE numerically?

Try $Y(t) = u(t, X(t))$. By Itô's formula, we get

$$\begin{aligned} dY(t) &= \partial_t u dt + \partial_x u dX(t) + \frac{1}{2}\partial_x^2 u (dX(t))^2 \\ &= \partial_t u dt + \partial_x u b dt + \partial_x u \sigma dW_t + \frac{1}{2}\partial_x^2 u \sigma^2 dt \\ &= \left(\partial_t u + b\partial_x u + \frac{1}{2}\sigma^2 \partial_x^2 u\right) dt + \partial_x u \sigma dW_t. \end{aligned}$$

Compare with FBSDE, we get

$$\begin{aligned} & h(t, X(t), u(t, X(t)), Z(t)) \\ = \; & \partial_t u(t, X(t)) + b(t, X(t), u(t, X(t)), Z(t)) \partial_x u(t, X(t)) \\ & + \frac{1}{2}\sigma^2(t, X(t), u(t, X(t)), Z(t)) \partial_x^2 u(t, X(t)) \end{aligned} \tag{5.6}$$

and

$$Z(t) = \sigma(t, X(t), u(t, X(t)), Z(t)) \partial_x u(t, X(t)).$$

Four step scheme (Ma-Protter-Yong):
Step 1: Find function $z(t, x, y, p)$ satisfying

$$z(t, x, y, p) = \sigma(t, x, y, z(t, x, y, p))p. \tag{5.7}$$

Step 2: Solve PDE

$$\begin{cases} \partial_t u + b(t, x, u, z(t, x, u, \partial_x u))\partial_x u \\ +\frac{1}{2}\sigma^2(t, x, u, z(t, x, u, \partial_x u))\partial_x^2 u \\ = h(t, x, u, z(t, x, u, \partial_x u)) \\ u(T, x) = g(x), \qquad x \in \mathbb{R}. \end{cases} \tag{5.8}$$

Step 3: Solve SDE

$$dX(t) = \tilde{b}(t, X(t))dt + \tilde{\sigma}(t, X(t))dW_t, \quad X_0 = x, \tag{5.9}$$

where

$$\tilde{b}(t, x) = b(t, x, u(t, x), z(t, x, u, \partial_x u(t, x))).$$

Step 4:

$$\begin{cases} Y(t) &= u(t, X(t)) \\ Z(t) &= z(t, X(t), Y(t), \partial_x u(t, X(t))). \end{cases} \tag{5.10}$$

(A1) The generator g has the following form: for $z = (z_1, \cdots, z_l)$,

$$g(x, y, z) = C(x, y)y + \sum_{j=1}^{l} D_j(x, y)z_j,$$

and $b(x, y)$, $\sigma(x)$, $g(x, y, z)$, $f(x)$, $C(x, y)$ and $D(x, y)$ are all bounded and Lipschitz continuous maps with bounded partial derivatives up to order 2. Furthermore, the matrix $\sigma\sigma^*$ is uniformly positive definite, and the function f is integrable. Here σ^* denote the transpose of the matrix σ.

We consider the following FBSDE in the fixed duration $[0, T]$:

$$\begin{cases} dX(t) = b\left(X\left(t\right), Y\left(t\right)\right) dt + \sigma\left(X\left(t\right)\right) dW(t), \\ -dY(t) = g(X(t), Y(t), Z(t))dt - Z(t)dW(t), \\ X(0) = x, Y(T) = f(X(T)), \end{cases} \tag{5.11}$$

where $b : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times l}$,
$g : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times l} \to \mathbb{R}^k$ and $f : \mathbb{R}^d \to \mathbb{R}^k$.

The PDE becomes

$$\begin{cases} -\frac{\partial u(t,x)}{\partial t} & = & Lu(t,x) + C\left(x, u\left(t,x\right)\right) u\left(t,x\right) \\ & & + \sum_{j=1}^{l} \sigma_j\left(x\right) D_j(x, u\left(t,x\right))\partial_x u\left(t,x\right) \\ u(T,x) & = & f(x), \end{cases}$$

and

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_{x_i,x_j} + \sum_{i=1}^{d} b_i \partial_{x_i},$$

with $a_{ij} = \left(\sigma\sigma^T\right)_{ij}$, $\sigma = (\sigma_1, \cdots, \sigma_l)$ and b_i being the ith coordinate of b.

For $0 \leq t \leq T$, assume $v(t,x) = u(T-t,x)$. Then

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} & = & Lv(t,x) + C(x, v(t,x))\, v(t,x) \\ & & + \sum_{j=1}^{l} \sigma_j(x)\, D_j(x, v(t,x)) \partial_x v(t,x) \\ v(0,x) & = & f(x). \end{cases} \tag{5.12}$$

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PDE (5.12) can be rewritten as

$$\begin{aligned}\frac{\partial v(t,x)}{\partial t} &= \frac{1}{2}\sum_{i,j=1}^{d}\partial_{x_i,x_j}\left(a_{ij}(x)v(t,x)\right) \\ &\quad -\sum_{i=1}^{d}\partial_{x_i}\left(\tilde{b}_i\left(x,v\right)v\left(t,x\right)\right) \\ &\quad +\tilde{c}\left(x,v\left(t,x\right)\right)v\left(t,x\right)\end{aligned}$$

where

$$\begin{aligned}\tilde{c}\left(x,v\right) &= C\left(x,v\right)-\sum_{i=1}^{d}\partial_{x_i}b_i\left(x,v\right)\\ &\quad -\sum_{i=1}^{d}\partial_{x_i}\tilde{D}_i\left(x,v\right)+\frac{1}{2}\sum_{i,j=1}^{d}\partial_{x_i,x_j}a_{ij}(x),\end{aligned}$$

$$\tilde{b}_i\left(x,v\right)=\sum_{j=1}^{d}\partial_{x_j}a_{ij}(x)-b_i(x,v)-\tilde{D}_i\left(x,v\right),$$

and

$$\tilde{D}_i\left(x,v\right)=\sum_{j=1}^{l}D_j\left(x,v\right)\sigma_{ij}\left(x\right).$$

Theorem (Extension of Kurtz-X. (1999))

For $0 < t \leq T$, $i = 1, 2, \cdots$, *let*

$$\begin{cases} dX_i(t) = \tilde{b}(X_i(t), v(t, X_i(t)))dt + \sigma(X_i(t))dB_i(t), \\ dA_i(t) = A_i(t)\tilde{c}\left(X_i(t), v\left(t, X_i(t)\right)\right) dt \\ V(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} A_j(t)\delta_{X_j(t)} \end{cases} \tag{5.13}$$

with i.i.d initial random sequence $\{(X_i(0), A_i(0)), i \in \mathbb{N}\}$ *taking values in* $\mathbb{R}^d \times \mathbb{R}$, *where* $\{B_i(t), i \in \mathbb{N}\}$ *are independent standard Brownian motions,* $V(t)$ *has density* $v(t, x)$ *and* $v(0, x) = f(x)$. *Then,* $v(t, x)$ *solves SPDE (5.12).*

6. Application to superprocesses

At time $t = 0$, k_n particles at locations

$$x_i^n \in \mathbb{R}, \qquad i = 1, 2, \cdots, k_n.$$

- Each has $expo.(n)$ clock (indep. from each other).

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- When time-up, the particle will split/die w/ equal prob.
- Denote particle by a multi-index α.
- Between branching times, particle α moves according to

$$dx_t^\alpha = b(x_t^\alpha)dt + c(x_t^\alpha)dW_t + e(x_t^\alpha)dB_t^\alpha$$

W, B^α indep. B.M.

Define measure-valued process

$$\mu_t^n = \frac{1}{n} \sum_{\alpha \sim t} \delta_{x_\alpha^n(t)}$$

where $\alpha \sim t$ means particle α is alive at time t.

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Convergence Theorem

$(\mu_t^n) \Rightarrow (\mu_t)$ unique sol. to MP: μ_t is $\mathcal{M}(R)$-valued

$$M_t(f) \equiv \langle \mu_t, f \rangle - \langle \mu, f \rangle - \int_0^t \langle \mu_s, bf' + af'' \rangle \, ds$$

continuous martingale with

$$\langle M(f) \rangle_t = \int_0^t \left(\langle \mu_s, f^2 \rangle + \langle \mu_s, cf' \rangle^2 \right) ds$$

where $a(x) = \frac{1}{2}(e(x)^2 + c(x)^2)$.

Studied by Adler and Skoulakis (2001) and Wang (1998) among others.

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(CMP)

$$\begin{aligned} N_t(f) &\equiv \langle \mu_t, f \rangle - \langle \mu, f \rangle - \int_0^t \langle \mu_s, bf' + df'' \rangle \, ds \\ &\quad - \int_0^t \langle \mu_s, cf' \rangle \, dW_s \end{aligned}$$

P^W-martingale w/

$$\langle N(f) \rangle_t = \int_0^t \langle \mu_s, f^2 \rangle \, ds$$

Formally,

$$\begin{aligned} N_t(f) &= \langle \mu_t, f \rangle - \langle \mu, f \rangle \\ &\quad - \int_0^t \left\langle \mu_s, (b + c\dot{W}_s) f' + d f'' \right\rangle ds, \end{aligned}$$

Therefore,

$$E^W \exp\left(- \langle \mu_t, f \rangle\right) = \exp\left(- \langle \mu, y_{0,t} \rangle\right),$$

where

$$\begin{aligned} y_{s,t} &= f + \int_s^t c\partial_x y_{r,t} \dot{W}_r dr \\ &\quad + \int_s^t \left(b\partial_x y_{r,t} + a\partial_x^2 y_{r,t} - y_{r,t}(x)^2\right) dr. \end{aligned}$$

Stochastic integral is backward Itô integral.

Theorem (Xiong, 2004, AP)

$$\mathbb{E}^W \exp\left(-\langle \mu_t, f\rangle\right) = \exp\left(-\langle \mu, y_{0,t}\rangle\right),$$

where $Lf = af'' + bf'$,

$$\begin{aligned} y_{s,t} &= f + \int_s^t c\partial_x y_{r,t} \hat{d} W_r \\ &\quad + \int_s^t \left(Ly_{r,t} - y_{r,t}(x)^2\right) dr. \end{aligned}$$

Forward version:

$$\begin{aligned} v_t(x) &= f(x) + \int_0^t \left(L v_r(x) - \frac{\gamma}{2} v_r(x)^2 \right) dr \\ &\quad + \int_0^t \int_U h(y,x) \nabla v_r(x) W(dr dy). \end{aligned} \tag{6.14}$$

Theorem

i) The SLLE (6.14) has a unique solution $v_t(x)$.
ii) v_t *is the unique solution of the following infinite particle system:* $i = 1, 2, \cdots,$

$$\begin{aligned} d\xi_t^i &= dB_i(t) - \int_U h(y, \xi_t^i) W(dtdy) \\ &\quad + \left(2a'(\xi_t^i) - \int_U h(y, \xi_t^i) \nabla h(y, \xi_t^i) \mu(dy) \right) dt, \end{aligned} \tag{6.15}$$

$$dm_t^i = m_t^i \left(\left(a''(\xi_t^i) - v_t(\xi_t^i) \right) dt - \int_U \nabla h(y, \xi_t^i) W(dtdy) \right), \tag{6.16}$$

$$\nu_t = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n m_t^i \delta_{\xi_t^i}, \qquad a.s. \tag{6.17}$$

where for any $t \geq 0$, ν_t *is absolutely continuous with respect to Lebesgue measure with* v_t *as the Radon-Nikodym derivative.*

Thanks!